When Fair Betting Odds are not Degrees of Belief

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1. Introduction

The "Dutch Book" argument, tracing back to Ramsey (1926) and deFinetti (1974), offers prudential grounds for action in conformity with personal probability. Under several structural assumptions about combinations of stakes (that is, assumptions about the combination of wagers), your betting policy is consistent (*coherent*) only if your *fair-odds* are probabilities. The central question posed here is the following one: Besides providing an operational test of *coherent* betting, does the "Book" argument also provide for adequate measurement (*elicitation*) of the agent's degrees of beliefs? That is, are an agent's *fair odds* also his/her personal probabilities for those events?

We argue the answer is "No!" The problem is created by state-dependent utilities. The methods of elicitation proposed by Ramsey, by deFinetti and by Savage (1954), are inadequate to the challenge of state-dependent values.[1]

2. A review of the Dutch Book argument.

A bet on/against event E, at odds of r:(1-r) with total stake S > 0 (say, bets are in $ units), is specified by its payoffs, as follows.[2]

	E	**-E**
bet on E	win $(1-r)S	lose $rS
bet against E	lose $(1-r)S	win $rS
abstain from betting	status quo	status quo

We assume that the *status quo* (the consequence of abstaining) represents no net change in wealth for the agent. It is depicted by a 0 payoff in the units of the stake. The "Book" argument depends upon four other structural assumptions (**a** — **d**) about the agent's preferences, listed below[3] Several technical terms facilitate the statement of those assumptions.

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Let *strict preference* be denoted by $<$ and let *indifference* be denoted by $\approx$. Call an option (act) **A** *favorable* in case it is strictly preferred to abstaining (abstaining $<$ **A**), call it *unfavorable* in case abstaining is strictly preferred (**A** $<$ abstaining), and call it *fair* whenever the agent is indifferent between it and abstaining (abstaining $\approx$ **A**). Last, say that option $\mathbf{A}_2$ *dominates* option $\mathbf{A}_1$ provided there is some (finite) partition π of events, $\pi = \{E_1, ..., E_n\}$, where event-by-event $\mathbf{A}_2$ yields a larger payoff than $\mathbf{A}_1$. Then:

(**a**) Given an event E, a betting rate r:(1-r) and stake S, the agent's preferences satisfy exactly one of these three profiles.

(**a.1**) betting on $<$ abstaining $<$ betting against E,

or (**a.2**) betting on $\approx$ abstaining $\approx$ betting against E,

or (**a.3**) betting against E $<$ abstaining $<$ betting on E.

(**b**) The (finite) conjunction of favorable bets is a favorable option, the (finite) conjunction of unfavorable bets is unfavorable, and the (finite) conjunction of fair bets is fair.

(**c**) For each event E there exists a stake S and *fair* betting rate r_E, i.e., where pro file (**a.2**) holds.

(**d**) The agent strictly prefers dominating options.[4]

Dutch Book Theorem: If the structural assumptions obtain, fair betting odds are probabilities.

Proof: See deFinetti (1974), or Shimony (1955).

Placing the emphasis differently we have, if the agent offers fair odds which are ***not*** probabilities (and his/her preferences satisfy the two structural assumptions **a** and **b**), then there is some favorable combination of bets which is dominated by abstaining. That is, then the agent's preferences are *incoherent* (in deFinetti's sense): there exists some so-called *favorable* combination of bets for which the agent loses in each event in a finite partition.

Suppose the agent's preferences satisfy the structural assumptions (**a**)—(**d**) when payoffs are in monetary amounts. Do the agent's fair odds reflect his/her personal probabilities for the states? That is, do fair odds operationalize degrees of belief? In the next two sections we explain why not. We argue that this betting argument does not give a satisfactory reduction of personal probability to coherent preference.

3. The Dollar-Yen problem

For simplicity, let π be a three state partition: $\pi = \{E_1, E_2, E_3\}$. The three states are pairwise disjoint and mutually exhaustive. Suppose that Smith's preferences over bets in dollars are coherent and satisfy the structural assumptions for the Dutch Book argument. For each state, let Smith's fair-odds, r_{Ei} (i = 1, 2, 3), be the same ratio, 1: 2. That is, Smith is indifferent between betting on or against E_i when 1/3 the stake is on E_i. Then, Smith is indifferent among the following three (favorable) acts:

	E_1	E_2	E_3
A_1	\$1	\$0	\$0
A_2	\$0	\$1	\$0
A_3	\$0	\$0	\$1

$$A_1 \approx A_2 \approx A_3$$

It is widely assumed (Ramsey, deFinetti, Savage, etc.) that these preferences show that Smith's personal probability for the three states is the uniform distribution: $P(E_i) = 1/3$.

Next, we offer Smith bets on the same three states, this time with monetary payoffs in (Japanese) Yen. Again, suppose Smith's preferences are coherent and satisfy the structural assumptions on bets . However, Smith's fair-odds (with stakes in Yen) lead to indifference among the following three acts A_i* (i = 1, 2, 3),

	E_1	E_2	E_3
A_1^*	100 Yen	0 Yen	0 Yen
A_2^*	0 Yen	125 Yen	0 Yen
A_3^*	0 Yen	0 Yen	150 Yen

$$A^*_1 \approx A^*_2 \approx A^*_3$$

From these indifferences, we recover a different personal probability: $P^*(E_1) = 15/37$; $P^*(E_2) = 12/37$; and $P^*(E_3) = 10/37$.

If we accept the usual interpretation of fair odds as personal probabilities, then we believe that Smith cannot be rational and hold both sets of preferences. We are encouraged to believe that a rational Smith cannot be indifferent among the preceding three dollar-acts and also be indifferent among the three Yen-acts.

That belief is mistaken! Let the states indicate the rate of exchange between the two currencies, as denoted by ≡. When state E_1 obtains, \$1 may be (freely) exchanged for 100 Y, \$1 ≡ 100Y; when E_2 obtains, \$1 ≡ 125Y; and when E_3 obtains, \$1 ≡ 150 Y. Then $A_i \approx A_i^*$ (i =1,2,3), as the following table makes clear.

	100 Y/\$1	125 Y/\$1	150 Y/\$1
A_1^*	100 Yen	0 Yen	0 Yen
A_2^*	0 Yen	125 Yen	0 Yen
A_3^*	0 Yen	0 Yen	150 Yen

Smith has fair betting odds in terms of each currency, odds that satisfy the structural constraints (**a-d**). Nonetheless, at least one of these systems of fair odds does not accurately reflect his/her degrees of belief for the three states E_i. What is wrong with using fair odds to measure degrees of belief? How can we tell which (if either) system of betting rates indicate Smith's degrees of belief over the three states? We address these questions in the next section.[5]

4. Diagnosis

With respect to a finite partition $\pi = \{E_1, ..., E_n\}$, let us represent the act A_i as a function from states to outcomes: $A_i(E_j) = o_{ij}$.

The canonical decision matrix: acts ⊗states

	E_1	E_2			E_j			E_n
A_1	O_{11}	O_{12}			O_{1j}			O_{1n}
A_2	O_{21}	O_{22}			O_{2j}			O_{2n}
A_m	O_m	O_m			O_{mj}			O_{mn}

$A_i(E_j)$ = outcome o_{ij}.

An agent has a preference structure $<$ that conforms with subjective expected utility theory provided there exist:

(**1**) a personal probability $P_i(\bullet)$ over states, given act A_i,

(**2**) a (cardinal) utility $U_j(\bullet)$ over outcomes, given state E_j,

and (**3**) $\forall(A_1, A_2)\ A_1 < A_2$ iff $\Sigma_j P_1(E_j)U_j(o_{1j}) < \Sigma_j P_2(E_j)U_j(o_{2j})$.

The familiar expected utility theories of, e.g., Ramsey, deFinetti, Savage, and Anscombe and Aumann (1963), simplify (**1-3**) in two ways. First, they require that states be act-independent: $P_i(E_j) = P(E_j)$. The agent's personal probability for states is not a function which act is chosen. Second, they require that utility be state-independent: $U_j(o_{ij}) = U(o_{ij})$. The agent's utility for outcomes does not depend upon the states under which they occur. If $o_{ij} = o_{i'j'}$, then $U(o_{ij}) = U(o_{i'j'})$. Thus (**1-3**) become the conditions (**1*-3***) that there exist:

(**1***) a personal probability $P(\bullet)$ over states,

(**2***) a (cardinal) utility $U(\bullet)$ over outcomes,

and (**3***) $\forall(A_1, A_2)\ A_1 < A_2$ iff $\Sigma_j P(E_j)U(o_{1j}) < \Sigma_j P(E_j)U(o_{2j})$.

In the dollar-yen problem, we have constructed a partition by states which precludes a state-independent utility over both $ and Yen outcomes simultaneously. Nevertheless, Smith's preferences satisfy deFinetti's structural assumptions for bets, provided we limit the space of outcomes to one currency, or to the other. Using acts involving one currency only, we can give an expected utility representation for Smith's preferences in accord with (**1* - 3***). For acts with *dollar* outcomes, the unique representation (**3***) of Smith's preferences uses the uniform probability, $P(E_i) = 1/3$ $(i=1,2,3)$ and linear $U(\$k) = k$. For acts with *yen* outcomes, the unique representation (**3***) of Smith's preferences uses $P^*(E_1) = .4054$, $P^*(E_2) = .3243$, $P^*(E_3) = .2703$, and the linear utility $U(k\text{-yen}) = k$.

Of course, Smith's preferences for acts with payoffs in both currencies can agree with the expected utility theory of (**1-3**), even with the assumption of act-state independence (**1***). But, unfortunately, we can't tell from preferences over acts, alone, what is Smith's personal probability for states and utility for outcomes. In particular, we cannot tell either of these from knowing Smith's *fair odds*. Worse yet, the *underdetermination* is maximal, as the next result, shows.

Result. Suppose that the preference relation $<$ is represented by an expected (possibly, state-dependent) utility as follows: For probability P and utility U_j,

$$\forall(A_1,A_2)\ A_1 < A_2 \text{ iff } \Sigma_j P(E_j)U_j(o_{1j}) < \Sigma_j P(E_j)U_j(o_{2j}).$$

Let P^* be any other probability on π that agrees with P on null states, i.e., $P(E_j) > 0$ iff $P^*(E_j) > 0$. Let $c_j = P(E_j)/P^*(E_j)$ and define $U^*_j(\ ) = c_jU_j(\ )$. Then,

$$\forall(A_1,A_2)\ A_1 < A_2 \text{ iff } \Sigma_j P^*(E_j)U^*_j(o_{1j}) < \Sigma_j P^*(E_j)U^*_j(o_{2j}).$$[6]

Thus, for each coherent system of preferences $<$, the family of possible (act-independent) probability/utility pairs that agree with $<$ according to expected utility is constrained solely by probability-0 (null) states.

There is a heuristic analogy for this decision-theory problem. We are familiar with Poincaré's (1982, p. 88) parable about separating the *geometry* from the *physics* based on the qualitative relation, *is no longer than*. What shall count as a *rigid rod*? In decision theory, we face the problem of separating *utility* from *probability* based on the qualitative relation, *is preferred to*, ($<$). The familiar approach (deFinetti, Savage, etc.), which aims at a representation in the form (**3***), directs us to find a pair of outcomes that serve (analogously) as *rigid Utility-rods* across π: outcomes whose values are state-independent.[7]

For a typical system of preferences over acts, there are many choices of what outcomes *may* carry state-independent utility. Different choices here yield different probability/utility pairs for representing the same preference relation over acts. In the Dollar—Yen example, the two rival representations arise from switching between taking \$-payoffs as state-independent in utility and taking Yen-payoffs that way.

The matter worsens in practice. If elicitation of personal probability from preference is made felicitous by using acts defined with a very few outcomes, then the lack of robustness in the resulting representation is magnified. It is easy for different elicitors to report inconsistent attributions of probability to the same agent when, in accord with the preference-data, they assume different outcomes have state-independent utility. If, in the Dollar-Yen example, an elicitor is lucky enough to include acts with both currencies in the preference questionnaire put to Smith, it will be discovered that Smith has a state-dependent utility for money. The elicitor remains in the dark, however, about what Smith's personal probability really is. No additional betting information will help out.

In a recent essay (1990), we develop this theme in connection with Anscombe & Aumann's "horse lottery" theory, and show that it surfaces in Savage's problem of "small worlds" (1954, pp. 82-91). We simplify some ideas of Karni, Schmeidler and Vind (1983) concerning the extra information about preferences that suffice to resolve the under-determination posed by the preceding **Result**. The new "data" on preferences make contrasts outside the original space of acts. It remains to be seen how these considerations can be used to improve elicitation of degrees of belief in deFinetti's setting of coherent bets.

Notes

[1]Here, we do not consider other approaches for eliciting degrees of belief, e.g., using qualitative probabilities: where the agent provides an ordering of events under the binary relation, *is more probable than*. See Narens (1980) for a helpful discussion of that.

[2]By permitting $S < 0$ (negative stakes), we can reverse betting "on" and betting "against."

[3]The structural assumptions are not obligatory, as Schick (1986) explains. Kadane and Winkler (1988) explore a market's effects on an agent's fair odds when there is decreasing marginal utility. For a modified Dutch Book theorem, with a more liberal version of the first assumption, see Seidenfeld,T., Schervish, M.J., and Kadane, J.B. (1990).

[4]It follows from these assumptions that the agent's attitude towards a simple bet is independent of the size of the stake. That is, each event E carries a unique *fair odds*

r_E for betting on E, independent of S. For betting odds greater than r_E, the agent prefers betting against E (profile **a.1**), at the fair odds the agent is indifferent (profile **a.2**), and with betting odds less than r_E the agent prefers to bet on E (profile **a.3**).

[5]Of course, Smith (an expected utility maximizer) has a personal rate of exchange between dollars and yen, $1 $\equiv_{Smith}$ 121.62 Yen. That is, Smith is indifferent between $1 outright and 121.62 Yen outright. These are his/her marginal expectations for the two currencies.

[6]The proof is trivial. This result has been pointed out in other contexts, for example: Arrow (1974), Karni, Schmeidler, and Vind (1983), and Rubin (1987).

[7]It is ironic that deFinetti writes that, for his Dutch Book argument, the outcomes for bets are required to be "rigid" in the sense that their utility is linear in quantity (1974, p. 77). That feature is secured by conditions (**a-d**) on betting. However, there is another sense of "rigidity" of outcome, also required for deFinetti's argument: state-independent utility. Unfortunately, that sense cannot be explicated in the language of bets, as the **Result** shows.

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